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THE SLOW ASYMMETRIC MOTION OF TWO DROPS IN A VISCOUS MEDIUM *

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An exact solution of Stokes equations is derived in the case of two spherical drops moving in a viscous medium at velocities normal to their line of centers. Results of numerical calculation of hydrodynamic forces are presented, and the passing to the limits of published solutions discussed. The behavior of hydrodynamic forces is investigated in the case of contacting spheres.

A considerable number of publications deals with the problem of finding the exact solution of Stokes equations in the case of motion of two solid spherical particles in a viscous medium. The solution in /1/ relates to slow rotation of particles about their line of centers, while translational motion of spheres along the line of their centers was considered in /2—4/. These solutions were obtained in bispherical coordinates and made possible the representation of hydrodynamic forces in the form of infinite series whose common terms have analytic expressions. A method of exact solution of Stokes equations in the asymmetric case when solid spheres translate along or rotate about axes normal to their line of centers was developed in /5,6/. Hydrodynamic forces in that case can also be represented by infinite series, but their common terms cannot be determined explicitly, and have to be obtained by solving a system of difference equations. The most comprehensive analysis of the asymmetric problem was given in /7—9/ which contain numerical data and an extensive bibliography. Owing to the linearity of the Stokes problem, the results of /1—9/ make it possible to calculate the interaction of two solid spheres in any arbitrary motion.

An exact solution of the axisymmetric problem of fluid spheres moving along their line of centres was obtained in /10,11/. In the present paper the exact solution is derived for the asymmetric case in which drops move at velocities normal to the line of their centers. Owing to the problem linearity the proposed solution together with /10,11/ makes possible the calculation of the interaction between drops in arbitrary motion.

An attempt was made in /12/ to obtain an asymptotic solution of the considered problem by the method of imaging that is applicable in the case of considerable distances between the sphere surfaces. It is shown below that the solution contains errors.

1. Statement of the problem. The exact solution of Stokes equations.

We consider fluid sphere of radius a_1 and a_2, with dynamic viscosities μ_1 and μ_2 moving normally to their line of centers at velocities $\mathbf{V}_1$ and $\mathbf{V}_2$ in a medium of viscosity μ_e. The Reynolds numbers are assumed low, and the problem is investigated in the Stokes approximation. As the boundary conditions we take the absence of flow through contact surfaces, the continuity of velocity and tangential stress at the sphere surfaces, and the fluid quiescence at infinity. Surface tension at the interface of fluids is assumed fairly high so that the deviation of particle shape from spherical can be neglected, and it is not necessary to take into account the boundary condition for continuity of normal stresses.

Let us consider the Cartesian system of coordinates (x, y, z) and the connected with it system of cylindrical coordinates (r, θ, z), with the z-axis on the line of centers O_2O_1 (Fig.1). The half-plane $y = 0$, $x > 0$ corresponds to $\theta = 0$. Without loss of generality, we assume that in the Cartesian coordinate system $\mathbf{V}_i = (\delta_i V, 0, 0)$, where $\delta_i = 0$ or $\delta_i = 1$ $(i = 1, 2)$. The solution structure is assumed to be the same as in /5—8/ and the fluid velocity components in the free flow regions denoted in Fig.1 by 1, 2, and e are sought in the form

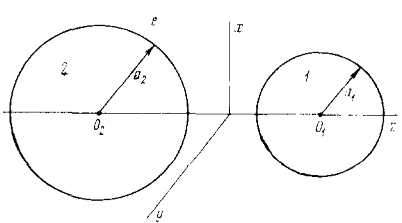

Fig.1

$$v_r = V(rF/c + \chi + \psi)\cos\theta \tag{1.1}$$
$$v_\theta = V(\chi - \psi)\sin\theta, \quad v_z = V(zF/c + 2\Phi)\cos\theta$$

where parameter c of dimension length is determined in (1.3) below. The unknown functions F, χ, ψ and Φ satisfy the equations

$$L_1F = L_1\Phi = L_2\chi = L_0\psi = 0 \tag{1.2}$$
$$L_m = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + \partial^2/\partial Z^2 - m^2r^{-2}$$

As shown in /8/, the Stokes equation $\mathrm{rot}\,(\Delta \mathbf{v}) = 0$ is in this case identically satisfied.

Using the bispherical system of coordinates

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$$z = \frac{c\,\mathrm{sh}\,\eta}{\mathrm{ch}\,\eta - \mu}, \quad r = \frac{c \sin\xi}{\mathrm{ch}\,\eta - \mu}, \quad \mu = \cos\xi$$

It is possible to determine parameter c and the quantities $\eta_1 > 0$ and $\eta_2 < 0$ so that the sphere of radius a_i is the coordinate surface $\eta = \eta_i = \mathrm{const}$ by setting

$$\mathrm{ch}\,\eta_1 = \frac{(1+\varepsilon)(1+k)+k\varepsilon^2/2}{1+k+k\varepsilon}, \quad \mathrm{sh}\,\eta_2 = -k\,\mathrm{sh}\,\eta_1, \quad c = a_1\,\mathrm{sh}\,\eta_1 \tag{1.3}$$

where εa_1 is the gap between spheres and $k = a_1 / a_2$.

As shown in /8/, solutions of Eqs. (1.2) can be sought in the form

$$F = \zeta \sin\xi \sum_{n=1}^{\infty} f_n(\eta) P_n'(\mu), \quad \Phi = \zeta\sin\xi \sum_{n=1}^{\infty} \varphi_n(\eta) P_n'(\mu) \tag{1.4}$$

$$\chi = \zeta \sin^2\xi \sum_{n=2}^{\infty} \chi_n(\eta) P_n''(\mu), \quad \Psi = \zeta \sum_{n=0}^{\infty} \psi_n(\eta) P_n(\mu)$$

$$\zeta = (\mathrm{ch}\,\eta - \mu)^{1/2}$$

where $P_n(\mu)$ is a Legendre polynomial of n-th power and f_n, φ_n, χ_n and ψ_n are linear combinations of functions $\exp[(n+1/2)\eta]$ and $\exp[-(n+1/2)\eta]$. Using the transform

$$\begin{aligned} \alpha_n &= 5f_n + 2\psi_n - 2(n-1)(n+2)\chi_n \\ \beta_{n-1} &= -(n-1)f_{n-1} - \psi_{n-1} + (n-2)(n-1)\chi_{n-1} \\ \gamma_{n+1} &= (n+2)f_{n+1} - \psi_{n+1} + (n+2)(n+3)\chi_{n+1} \quad (n \geqslant 1) \end{aligned} \tag{1.5}$$

we introduce in addition to f_n, φ_n, χ_n and ψ_n functions $\alpha_n(\eta)$, $\beta_n(\eta)$ and $\gamma_n(\eta)$.

To eliminate singularities of the velocity field at points $z = \pm c$, $r = 0$ that lie inside the spheres, we set

$$\begin{Vmatrix} \alpha_n^1 \\ \beta_n^1 \\ \gamma_n^1 \\ \varphi_n^1 \end{Vmatrix} = \begin{Vmatrix} I_n^1 \\ K_n^1 \\ M_n^1 \\ A_n^1 \end{Vmatrix} \exp[-(n+1/2)\eta], \quad \begin{Vmatrix} \alpha_n^2 \\ \beta_n^2 \\ \gamma_n^2 \\ \varphi_n^2 \end{Vmatrix} = \begin{Vmatrix} J_n^2 \\ L_n^2 \\ N_n^2 \\ B_n^2 \end{Vmatrix} \exp[(n+1/2)\eta] \tag{1.6}$$

In the external region

$$\begin{Vmatrix} \alpha_n^e \\ \beta_n^e \\ \gamma_n^e \\ \varphi_n^e \end{Vmatrix} = \begin{Vmatrix} I_n^e \\ K_n^e \\ M_n^e \\ A_n^e \end{Vmatrix} \exp[-(n+1/2)\eta] + \begin{Vmatrix} J_n^e \\ L_n^e \\ N_n^e \\ B_n^e \end{Vmatrix} \exp[(n+1/2)\eta] \tag{1.7}$$

where the superscripts e and i $(i = 1, 2)$ denote quantities in respective regions shown in Fig.1.

Using the data in /8/ we find that the equation $\mathrm{div}\,\mathbf{v} = 0$ of continuity in the external region is equivalent to two relations

$$\begin{aligned} & I_n^e + K_{n-1}^e + M_{n+1}^e - 2(2n+1)A_n^e + 2(n-1)A_{n-1}^e + \\ & \quad 2(n+2)A_{n+1}^e = 0 \\ & J_n^e + L_{n-1}^e + N_{n+1}^e + 2(2n+1)B_n^e - 2(n-1)B_{n-1}^e - \\ & \quad 2(n+2)B_{n+1}^e = 0 \quad (n \geqslant 1) \end{aligned} \tag{1.8}$$

For inner regions the continuity equations also reduce to two difference equations by substituting in the case of region 1 superscript unity for e throughout the first of Eqs. (1.8), and in region 2 substitute in the second of Eqs. (1.8) J_n^2 etc. for J_n^e etc.

The equations of fluid velocity continuity at the sphere surfaces yield

$$\begin{aligned} f_n^i - f_n^e &= \frac{2}{\mathrm{sh}\,\eta}\left[\frac{n-1}{2n-1} Z_{n-1}^i + \frac{n+2}{2n+3} Z_{n+1}^i - \mathrm{ch}\,\eta\, Z_n^i\right] \\ \psi_n^i - \psi_n^e &= \frac{1}{\mathrm{sh}\,\eta}\left[\frac{(n+1)(n+2)}{2n+3} Z_{n+1}^i - \frac{n(n-1)}{2n-1} Z_{n-1}^i\right] \\ \chi_n^i - \chi_n^e &= \frac{1}{\mathrm{sh}\,\eta}\left[\frac{1}{2n-1} Z_{n-1}^i - \frac{1}{2n+3} Z_{n+1}^i\right] \\ Z_n^i &= \varphi_n^i - \varphi_n^e, \quad \eta = \eta_i, \quad i = 1, 2 \end{aligned} \tag{1.9}$$

Using (1.5) — (1.9) we can reduce the solenoidality of velocity fields in the inner regions to the form

$$\begin{aligned} & I_n^e + 2(2n+1)A_n^e + \exp(2\eta_2)[K_{n-1}^e - 2(n-1)A_{n-1}^e] + \\ & \quad \exp(-2\eta_2)[M_{n+1}^e - 2(n+2)A_{n+1}^e] + \exp[(n+1/2)\eta_2]X_n^2 = 0 \\ & J_n^e - 2(2n+1)B_n^e + \exp(-2\eta_1)[L_{n-1}^e + 2(n-1)B_{n-1}^e] + \\ & \quad \exp(2\eta_1)[N_{n+1}^e + 2(n+2)B_{n+1}^e] + \exp[-(n+1/2)\eta_1]X_n^1 = 0 \quad (n \geqslant 1) \end{aligned} \tag{1.10}$$

$$X_n^i = \frac{4}{\operatorname{sh}\eta_i}\left[\frac{n-1}{2n-1}Z_{n-1}^i + \frac{n+2}{2n+3}Z_{n+1}^i + \left(\frac{|\operatorname{sh}\eta_i|}{2n+1} - \operatorname{ch}\eta_i\right)Z_n^i\right]$$
$$i = 1, 2$$

To simplify the writing of boundary conditions for impenetrability and of continuity of tangential stresses we introduce in the neighborhood of each surface the vector of relative fluid velocity $\mathbf{v}_* = \mathbf{v} - \mathbf{V}_i$ and the quantities

$$u_\eta = -\mu F \operatorname{sh}\eta + 2(1 - \mu\operatorname{ch}\eta)\Phi - (\chi + \psi - \delta_i)\sin\xi\operatorname{sh}\eta \tag{1.11}$$
$$u_\xi = -\frac{1}{\sin\xi}\left[(\operatorname{ch}\eta - \mu)F + \frac{2(\operatorname{ch}\eta-\mu)^2}{\operatorname{sh}\eta}\Phi + \frac{(\mu\operatorname{ch}\eta - 1)}{\operatorname{sh}\eta}u_\eta\right]$$

which differ from the respective covariant components of vector $\mathbf{v}_*$ by the coefficient $(cV\cos\theta)^{-1}$ $(\operatorname{ch}\eta - \mu)^2$. The impenetrability conditions $u_\eta^e = 0$ at the surface of spheres assume the form

$$\frac{\gamma_{n+1}^e}{2n+3} - \frac{\beta_{n-1}^e}{2n-1} + \frac{2\operatorname{ch}\eta}{\operatorname{sh}\eta}\left[\frac{n-1}{2n-1}\varphi_{n-1}^e + \frac{n+2}{2n+3}\varphi_{n+1}^e\right] - \frac{2\varphi_n^e}{\operatorname{sh}\eta} = \tag{1.12}$$
$$\delta_i\sqrt{2}\left\{\frac{\exp[-(n-1/2)|\eta|]}{2n-1} - \frac{\exp[-(n+3/2)|\eta|]}{2n+3}\right\}$$
$$n \geqslant 1,\ \eta = \eta_i,\ i = 1, 2$$

With allowance for (1.7) the equalities (1.8), (1.10), and (1.12) constitute a system of linear equations in I_n^e, J_n^e, K_{n-1}^e, L_{n-1}^e, M_{n+1}^e and N_{n+1}^e (this is the reason for using transform (1.5)). The solution of this system yields for the unknowns expressions in terms of A_m^e, B_m^e $(n-1 \leqslant m \leqslant n+1)$ and X_n^1 and X_n^2. The related formulas are very cumbersome and are not presented here.

The coefficients of all functions φ_n, f_n, χ_n and ψ_n may be expressed in terms of A_m^e, B_m^e, Z_m^1 and Z_m^2 $(n-2 \leqslant m \leqslant n+2)$ using (1.9) and the transform inverse of (1.5)

$$3(2n+1)f_n = (2n+1)\alpha_n + 2(n+2)\beta_n + 2(n-1)\gamma_n \tag{1.13}$$
$$6(2n+1)\chi_n = -(2n+1)\alpha_n - (2n+7)\beta_n - (2n-5)\gamma_n$$
$$6(2n+1)\psi_n = -(2n+1)\quad n(n+1)\alpha_n - (n+1)\quad(n+2)\times$$
$$(2n+3)\beta_n - n(n-1)(2n-1)\gamma_n$$

It remains to satisfy the boundary conditions of continuity of tangential stresses whose projections on ξ and θ are, respectively,

$$\partial u_\xi^e/\partial\eta = \lambda_i\partial u_\xi^i/\partial\eta \tag{1.14}$$
$$\frac{\partial}{\partial\eta}[(\operatorname{ch}\eta-\mu)(\chi^e - \psi^e + \delta_i)] = \lambda_i\frac{\partial}{\partial\eta}[(\operatorname{ch}\eta-\mu)(\chi^i - \psi^i + \delta_i)]$$
$$\lambda_i = \mu_i/\mu_e,\ \eta = \eta_i,\ i = 1, 2$$

Taking into account (1.11) and the impenetrability conditions, we represent the quantities $\partial u_\xi/\partial\eta$ when $\eta = \eta_i$ in the form

$$-\frac{3\operatorname{sh}\eta}{2\sin\xi}F - \frac{3(\operatorname{ch}\eta-\mu)}{\sin\xi}\Phi - \sin\xi\operatorname{ch}\eta\frac{DF}{\partial\eta} + \tag{1.15}$$
$$(\mu\operatorname{ch}\eta - 1)\frac{D}{\partial\eta}(\chi + \psi - \delta_i) - 2\operatorname{sh}\eta\sin\xi\frac{D\Phi}{\partial\eta}$$
$$D/\partial\eta = \partial/\partial\eta - 1/2\operatorname{sh}\eta(\operatorname{ch}\eta - \mu)^{-1}$$

Using (1.15), (1.4), (1.9), (1.5), and (1.13) we can represent the first of Eqs. (1.14) in the form of the difference equation

$$\frac{3(\lambda_i-1)}{1+\lambda_i}\left\{f_n^e\operatorname{sh}\eta + 2\left[\varphi_n^e\operatorname{ch}\eta - \frac{n-1}{2n-1}\varphi_{n-1}^e - \frac{n+2}{2n+3}\varphi_{n+1}^e\right]\right\} + \tag{1.16}$$
$$\operatorname{ch}\eta\left[\frac{n+2}{2n+3}(\gamma_{n+2}^\pm - \beta_n^\pm) + \frac{n-1}{2n-1}(\gamma_n^\pm - \beta_{n-2}^\pm) - (2n+1)f_n^\pm\right] +$$
$$(n+2)f_{n+1}^\pm + (n-1)f_{n-1}^\pm + \beta_{n-1}^\pm - \gamma_{n+1}^\pm +$$
$$2\operatorname{sh}\eta\left[\frac{(n+2)(n+3)}{2n+3}\varphi_{n+2}^\pm + \frac{(n-1)(n-2)}{2n-1}\varphi_{n-2}^\pm -\right.$$
$$\left.\frac{2n(n+1)(2n+1)}{(2n-1)(2n+3)}\varphi_n^\pm\right] +$$
$$\frac{4\lambda_i}{(1+\lambda_i)|\operatorname{sh}\eta|}\left\{\frac{(n-1)(n-2)}{(2n-1)(2n-3)}Z_{n-2}^i - \frac{2(n-1)^2\operatorname{ch}\eta}{(2n-1)^2}Z_{n-1}^i +\right.$$
$$\frac{2}{2n+1}\left[\frac{2n(n+1)}{(2n-1)(2n+3)} - 1\right]Z_n^i + \frac{2(n+2)^2\operatorname{ch}\eta}{(2n+3)^2}Z_{n+1}^i -$$
$$\left.\frac{(n+2)(n+3)}{(2n+3)(2n+5)}Z_{n+2}^i\right\} =$$

$$\frac{2^{3/2}\delta_i \operatorname{sh}\eta(\lambda_i-1)}{1+\lambda_i}\left\{\operatorname{ch}\eta\left[\frac{n+2}{2n+3}\exp(-(n+{}^3/_2)|\eta|)+\right.\right.$$
$$\left.\left.\frac{n-1}{2n-1}\exp(-(n-{}^1/_2)|\eta|)\right]-\exp(-(n+{}^1/_2)|\eta|)\right\}\quad(n\geqslant 1)$$
$$(1+\lambda_i)(n+{}^1/_2)f_n^{\pm}=df_n^e/d\eta\pm\lambda_i(n+{}^1/_2)f_n^e$$
$$\eta=\eta_i,\quad i=1,2$$

where the quantities $\beta_n^{\pm}$, $\gamma_n^{\pm}$ and $\varphi_n^{\pm}$ are determined similarly to $f_n^{\pm}$ using the upper sign for $i=1$ and the lower for $i=2$.

Using the second of Eqs. (1.14) we shall show that $f_n^{\pm}$ can be expressed in terms of A_m^e, B_m^e, Z_m^1 and Z_m^2 for $n-1\leqslant m\leqslant n+1$. Substitution of corresponding expressions for $f_{n-1}^{\pm}$ and $f_{n+1}^{\pm}$ in (1.16) makes it possible to obtain two fourth-order difference equations in A_n^e, B_n^e, Z_n^1 and Z_n^2.

The expressions for $(\operatorname{ch}\eta-\mu)(\chi-\psi+\delta_i)$ in the continuity equation $\operatorname{div}\mathbf{v}_*=0$ for the neighborhood of the sphere of radius a_i in bispherical coordinates is obtained from (1.14) in the form

$$\left[\frac{\partial^2}{\partial\eta^2}-\frac{3\operatorname{sh}\eta}{\operatorname{ch}\eta-\mu}\frac{\partial}{\partial\eta}\right](u_\eta^e-\lambda_i u_\eta^i)+ \tag{1.17}$$
$$\left[\frac{3\sin\xi\operatorname{sh}\eta}{(\operatorname{ch}\eta-\mu)^2}-\frac{2\sin\xi}{\operatorname{ch}\eta-\mu}\frac{\partial}{\partial\eta}\right](u_\xi^e-\lambda_i u_\xi^i)=0,\quad \eta=\eta_i,\quad i=1,2$$

which with allowance for (1.11) and the impenetrability conditions can be represented in the form

$$\left[2\frac{D}{\partial\eta}-\mu\operatorname{sh}\eta\frac{D^2}{\partial\eta^2}\right](F^e-\lambda_iF^i)+\left[2(1-\mu\operatorname{ch}\eta)\frac{D^2}{\partial\eta^2}+\right. \tag{1.18}$$
$$\left.2\right](\Phi^e-\lambda_i\Phi^i)-\sin\xi\operatorname{sh}\eta\frac{D^2}{\partial\eta^2}[(\chi^e+\psi^e-\delta_i)-\lambda_i(\chi^i+\psi^i-\delta_i)]=$$
$$=0,\quad \eta=\eta_i,\quad i=1,2$$

Using (1.4), (1.5), (1.9), and (1.12) we represent (1.18) in the form of the difference equation

$$f_n^{\pm}=\frac{2(1-\lambda_i)}{1+\lambda_i}\left\{\frac{\operatorname{sh}\eta}{2n+3}[\delta_i\sqrt{2}\exp(-(n+{}^3/_2)|\eta|)+\gamma_{n+1}^e]+\right. \tag{1.19}$$
$$\left.\frac{2(n+2)\operatorname{ch}\eta}{2n+3}\varphi_{n+1}^e-\varphi_n^e\right\}\mp\frac{\lambda_i}{2(1+\lambda_i)}X_n^i\quad(n\geqslant 1)$$
$$\eta=\eta_i,\ i=1,\ 2$$

where the upper sign is taken for $i=1$ and the lower for $i=2$.

The obtained representation of $f_n^{\pm}$ together with the expressions for coefficients I_n^e, J_n^e, K_{n-1}^e, L_{n-1}^e, M_{n+1}^e and N_{n+1}^e in terms of A_m^e and B_m^e $(n-1\leqslant m\leqslant n+1)$ and X_n^1 and X_n^2 makes it possible to obtain from (1.16) two fourth order difference equations in A_n^e, B_n^e, Z_n^1 and Z_n^2. The second pair of fourth order equations is obtained by using for $f_n^{\pm}$ another expression based on the first of equalities (1.13).

Remark. An attempt at direct reduction of condition (1.14) to four difference equations in A_n^e, B_n^e, Z_n^1 and Z_n^2 results in two of these containing differences of the sixth order and two of the eighth order.

The final system of equations is of the form

$$\sum_{k=-2}^{2}T_n^k\mathbf{w}_{n+k}=\delta_1\mathbf{S}_n^1+\delta_2\mathbf{S}_n^2,\quad n\geqslant 1 \tag{1.20}$$
$$T_n^k=0,\quad n+k<1$$

where $\mathbf{w}_n$ is a column vector with components

$$A_n^e\exp[-(n+{}^1/_2)\eta_2],\ B_n^e\exp[(n+{}^1/_2)\eta_1],\ Z_n^1,\ Z_n^2$$

Since the derivation of analytic expressions for elements of matrices T_n^k of the fourth order and of vectors $\mathbf{S}_n^j$ would be extremely laborious, a subroutine was developed for the computer which made possible the numerical determination of these for any n in accordance with the expounded scheme.

As the necessary condition of flow regularity up to the surface of spheres we specify that the solution $\mathbf{w}_n$ of system (1.20) must approach zero as $n\to\infty$.

When $\mathbf{w}_n$ is known it is possible to determine all functions φ_n, f_n, χ_n and ψ_n.

2. Calculation of hydrodynamic interaction of sphere. Note that all moments of hydrodynamic forces about the particle centers of mass are always zero. The moment of forces acting on a sphere of radius a_i is equal

$$\int_{S_i}(\mathbf{R}\times\mathbf{p}_n^e)\,dS \tag{2.1}$$

where S_i is the sphere surface area, $\mathbf{R}$ is the vector from the sphere center to the current point of its surface, and $\mathbf{p}_n$ is the stress vector. Taking into account the collinearity of vector $\mathbf{R}$ and the vector of the normal and the conditions of continuity of tangential stresses, vector $\mathbf{p}_n^i$ can be substituted in (2.1) for vector $\mathbf{p}_n^e$ and, then, use the Gauss theorem and the nondivergence of the stress tensor that follows from Stokes equations.

It follows from (1.1) that only the x-components of forces can be nonzero. We set

$$f_n^e(\eta) = C_n^e \exp[(n+1/2)\eta] + D_n^e \exp[-(n+1/2)\eta]$$

As shown in /8/, formulas

$$F_1^x = -\pi\mu_e a_1 V \operatorname{sh}\eta_1 8\sqrt{2}\sum_{n=1}^{\infty} n(n+1) C_n^e \tag{2.2}$$

$$F_2^x = \pi\mu_e a_2 V \operatorname{sh}\eta_2 8\sqrt{2}\sum_{n=1}^{\infty} n(n+1) D_n^e$$

are valid for the x-components of forces acting on particles, independently of boundary conditions at the sphere surfaces.

The sums of series (2.2) were computed as follows. From the two equations (1.19) we obtain

$$n(n+1)\left\|\begin{matrix} C_n^e \\ D_n^e \end{matrix}\right\| = \sum_{j=-1}^{1} R_n^j \mathbf{w}_{n+j} + \sum_{j=1}^{2} \delta_j \mathbf{G}_n^j, \quad n \geqslant 1 \tag{2.3}$$

Matrices R_n^j and vectors $\mathbf{G}_n^j$ were determined numerically. Then, applying to system (1.20) the method of matrix run-through, it is possible to express $\mathbf{w}_n$ in terms of $\mathbf{w}_{n+1}$ and $\mathbf{w}_{n+2}$ (the initial values of running coefficients are determined by the last of conditions (1.20)).Hence, using (2.3), we can write

$$\sum_{m=1}^{n-1} m(m+1)\left\|\begin{matrix} C_m^e \\ D_m^e \end{matrix}\right\| = Q_n^\circ \mathbf{w}_n + Q_n^{-1}\mathbf{w}_{n-1} + \delta_1 \mathbf{H}_n^1 + \delta_2 \mathbf{H}_n^2$$

Matrices Q_n^k and vectors $\mathbf{H}_n^j$ are determined recurrently. Numerical computations show that always $Q_n^\circ, Q_n^{-1} \to 0$ as $n \to \infty$. Since $\mathbf{w}_n \to 0$, hence for calculating forces it is sufficient to determine the limits of $\mathbf{H}_n^1$ and $\mathbf{H}_n^2$ as $n \to \infty$.

We stress that only a straight through run is used with this algorithm.

In the case of arbitrary motion of the spheres at velocities $\mathbf{V}_1$ and $\mathbf{V}_2$ normal to the line of centers we represent the hydrodynamic forces as follows:

$$\mathbf{F}_1 = -6\pi\mu_e a_1 [\Lambda_{11}(\mathbf{V}_1 - \mathbf{V}_2) + \Lambda_{12}\mathbf{V}_2]$$

$$\mathbf{F}_2 = -6\pi\mu_e a_2 [\Lambda_{21}(\mathbf{V}_2 - \mathbf{V}_1) + \Lambda_{22}\mathbf{V}_2]$$

The dependence between coefficients of drag determined in /12/ is equivalent to the relation

$$\Lambda_{11} - k^{-1}\Lambda_{21} = \Lambda_{12} \tag{2.4}$$

The calculated coefficients Λ_{ij} for $k = 0.1$, 0.5, and 1.0 and various ε and λ ($\lambda_1 = \lambda_2 = \lambda$) are shown in Table 1, where for each pair of λ, ε a column of values of $\Lambda_{11}, \Lambda_{21}, \Lambda_{22}$ appears for $k = 0.1$ and 0.5, while for $k = 1.0$ the column contains only the values of $\Lambda_{11}, \Lambda_{21}$, since then $\Lambda_{22} = \Lambda_{11} - \Lambda_{21}$ for $k = 1$.

Calculations were carried out with doubled accuracy on a high-speed electronic computer. Calculations had shown that for small ε the error of calculations with standard accuracy was sometimes insufficient, particularly in the case of large λ. The proposed solution is generally not valid when $\lambda_1 = \lambda_2 = \infty$, since conditions (1.14) imply that for $\eta = \eta_i$ the tangent stresses are zero in inner regions. Hence system (1.20) has a nonunique solution because the solution of the homogeneous system corresponds to arbitrary rotation of solid spheres about axes that pass through /their/ centers and are parallel to the y-axis. However, the moment of forces remains zero at the limit $\lambda_1, \lambda_2 \to \infty$ and, consequently, for large but finite λ_1 and λ_2 the coefficients Λ_{ij} must be close to the corresponding values of Λ_{ij}^s for spheres moving with free rotation. The values of Λ_{ij}^s can be determined as follows. When a solid sphere translates at velocity $(V_i, 0, 0)$ and rotates at velocity $(0, \Omega_i, 0)$ (defined in a Cartesian coordinate system), then force F_i^x and the force moment G_i^y acting on the sphere may, according to /7, 8/, be represented in the form

$$F_i^x = 6\pi\mu_e a_i \sum_{j=1}^{2} (F_{ij}^t V_j + F_{ij}^r a_j \Omega_j)$$

$$G_i^y = 8\pi\mu_e a_i^2 \sum_{j=1}^{2} (G_{ij}^t V_j + G_{ij}^r a_j \Omega_j)$$

(in a somewhat different notation than in /7,8/). Expressing angular velocities in terms of translational ones and taking into account that the moments G_i^y are zero, we obtain Λ_{ij}^s as a function of sixteen coefficients F_{ml}^t, F_{ml}^r, G_{ml}^t and G_{ml}^r. For certain pairs of k, ε from the Table it is possible to calculate the values of Λ_{ij}^s using the values of F_{ij}^t, F_{ij}^r, G_{ij}^t and G_{ij}^r

Table 1

k	λ	ε=10.0	5.0	1.0	0.1	0.01	0.001
0.1	0.0	0.6703	0.6722	0.6617	0.6301	0.6224	0.6215
		0.0160	0.0210	0.0276	0.0284	0.0283	0.0283
		0.6545	0.6522	0.6506	0.6511	0.6512	0.6513
	10.0	0.9844	1.0036	1.1322	1.4441	1.6085	1.6470
		0.0365	0.0512	0.0840	0.1215	0.1384	0.1424
		0.9467	0.9446	0.9483	0.9511	0.9514	0.9514
	30.0	1.0052	1.0270	1.1848	1.6352	1.9860	2.1271
		0.0382	0.0538	0.0904	0.1424	0.1782	0.1925
		0.9656	0.9637	0.9681	0.9717	0.9723	0.9724
	10^7	1.0167	1.0402	1.2174	1.7806	2.3998	2.9497
		0.0391	0.0552	0.0944	0.1581	0.2211	0.2767
		0.9760	0.9741	0.9791	0.9832	0.9842	0.9848
0.5	0.0	0.6686	0.6719	0.6871	0.6955	0.6955	0.6954
		0.0257	0.0420	0.0859	0.1123	0.1157	0.1160
		0.6429	0.6299	0.6022	0.5904	0.5892	0.5891
	10.0	0.9759	0.9869	1.0625	1.2217	1.2854	1.2968
		0.0551	0.0919	0.2124	0.3416	0.3867	0.3875
		0.9208	0.8949	0.8431	0.8286	0.8303	0.8312
	30.0	0.9959	1.0076	1.0910	1.2897	1.3992	1.4316
		0.0574	0.0958	0.2234	0.3737	0.4365	0.4546
		0.9385	0.9117	0.8585	0.8442	0.8473	0.8496
	10^7	1.0068	1.0190	1.1071	1.3355	1.5018	1.6030
		0.0587	0.0980	0.2297	0.3954	0.4872	0.5413
		0.9482	0.9209	0.8669	0.8526	0.8569	0.8620
1.0	0.0	0.6678	0.6701	0.6856	0.7051	0.7083	0.7087
		0.0278	0.0479	0.1142	0.1677	0.1760	0.1769
	10.0	0.9733	0.9806	1.0411	1.1811	1.2361	1.2457
		0.0592	0.1032	0.2703	0.4787	0.5406	0.5503
	30.0	0.9931	1.0008	1.0660	1.2286	1.3104	1.3328
		0.0617	0.1075	0.2832	0.5158	0.6040	0.6260
	10^7	1.0040	1.0119	1.0799	1.2579	1.3696	1.4277
		0.0630	0.1099	0.2905	0.5394	0.6570	0.7136

tabulated in /7,8/. It appears that for the same k, ε and $\lambda = 10^7$ the values of Λ_{ij} in Table 1 coincide to a high degree of accuracy with the corresponding values of Λ_{ij}^s (for instance, the values of Λ_{11} and Λ_{11}^s differ by less than unity of the least significant decimal digit).Note that the method of calculating Λ_{ij}^s differs in principle from the proposed here method of calculating Λ_{ij}.

An asymptotic formula for F_1^x was obtained in /12/ by the method of imaging, which for $\lambda_1, \lambda_2 \to \infty$ becomes the approximate formula derived there for solid spheres moving normally to the line of centers without rotation. The authors of /12/ had erroneously assumed that this confirms the correctness of the solution derived by them. But, since $\Lambda_{ij} \to \Lambda_{ij}^s$, the correct asymptotic formula for F_1^x as λ_1, $\lambda_2 \to \infty$ must be in agreement with the known asymptotic solution for solid spheres moving with free rotation (see /13/). There is actually no such agreement, hence it is possible to assume the existence of an error in the formula in /12/. An exact calculation by the method used in /12/ yields

$$F_1^x \simeq -2\pi\mu_e a_1 p_1 \left\{V_1^x\left[1 + \frac{1}{16} p_1 p_2 \alpha_1 \alpha_2 + \frac{1}{8} p_1 q_2 \alpha_1 \alpha_2^3 + \frac{1}{8} q_1 p_2 \alpha_1^3 \alpha_2 + \frac{1}{256} p_1^2 p_2^2 \alpha_1^2 \alpha_2^2\right] - V_2^x\left[\frac{1}{4} p_2 \alpha_2 + \frac{1}{4} q_2 \alpha_2^3 + \frac{1}{4} q_1 p_1^{-1} p_2 \alpha_1^2 \alpha_2 + \frac{1}{64} p_1 p_2^2 \alpha_1 \alpha_2^2 + \frac{3}{64} q_1 p_2^2 \alpha_1^3 \alpha_2^2 + \frac{3}{64} p_1 p_2 q_2 \alpha_1 \alpha_2^4 + \frac{1}{1024} p_1^2 p_2^3 \alpha_1^2 \alpha_2^3\right]\right\} \quad (2.5)$$

$$p_i = (1 + \lambda_i)^{-1}(2 + 3\lambda_i), \quad q_i = (1 + \lambda_i)^{-1}\lambda_i, \quad \alpha_i = a_i / l$$

where l is the distance between spheres. The coefficients at $\alpha_1\alpha_2^3$, $\alpha_1\alpha_2^4$ and $\alpha_1^3\alpha_2^2$ were incorrectly determined in /12/. The comparison with numerical results shows that formula (2.5) is a correct asymptotic expansion of F_1^x as $\alpha_1, \alpha_2 \to 0$. For $k = 0.5$ and $\varepsilon = 10$ and all λ the values of coefficients of Λ_{ij} calculated by formula (2.5) coincide with the exact values within not less than the fourth significant digit. However in the case of nearly touching spheres formula (2.5) yields, as a rule, a considerable error, particularly for large λ_1 and λ_2. For example, for $k = 0.25$, $\varepsilon = 0.005$ and $\lambda = 30$ it follows from formula (2.5) that $\Lambda_{11} = 1.12$, $\Lambda_{21} = 0.20$ and $\Lambda_{22} = 0.921$, while the exact solution yields the following values: $\Lambda_{11} = 1.6371$, $\Lambda_{21} = 0.32021$ and $\Lambda_{22} = 0.92885$. Moreover, formula (2.5) does not show the nonmonotonic behavior of coefficients of Λ_{11} and Λ_{21} which is generally present at a change of ε (see Table 1).

3. The behavior of coefficients of Λ_{ij} in the case of touching spheres.

Let us consider the dissipative function E

$$\frac{E}{2} = \mu_e \int_{D_e} e^{jk} e_{jk} d\tau + \sum_{i=1}^{2} \mu_i \int_{D_i} e^{jk} e_{jk}\, d\tau$$

which corresponds to the motion of two spheres. In this formula D_e and D_i are the regions denoted in Fig.1 by the symbols e and i $(i = 1, 2)$, and e_{jk} are components of strain rate tensor. Boundary conditions on the surface of spheres make it possible to represent E as a quadratic form of velocities: $E = -(\mathbf{F}_1\mathbf{V}_1 + \mathbf{F}_2\mathbf{V}_2)$. The dissipative function E^s in the case of two solid spheres moving with rotation can be represented in exactly the same form. Taking into account the boundary conditions for the remainder $E^s - E$ (for the same ε, a_i, and $\mathbf{V}_i$) we obtain the formula

$$\frac{E^s - E}{2} = \mu_e \int_{D_e} \varepsilon^{jk} \varepsilon_{jk}\, d\tau + \sum_{i=1}^{2} \mu_i \int_{D_i} e^{jk} e_{jk}\, d\tau, \quad \varepsilon_{jk} = e_{jk}{}^s - e_{jk}$$

The components $e_{jk}{}^s$ of the deformation rate tensor relate to the motion of solid spheres. Taking into account (2.4) we can obtain, owing to the positive definiteness of the quadratic forms of E and $E^s - E$, the inequalities

$$0 < \Lambda_{11} < \Lambda_{11}{}^s,\ 0 < \Lambda_{22} + \Lambda_{21} < \Lambda_{22}{}^s + \Lambda_{21}{}^s \tag{3.1}$$
$$(\Lambda_{21})^2 < k\Lambda_{11}\ (\Lambda_{22} + \Lambda_{21})$$

The asymptotic formulas in /9/ for coefficients $F_{ij}{}^t$, $F_{ij}{}^r$, $G_{ij}{}^t$, and $G_{ij}{}^r$ imply that the values of $\Lambda_{ij}{}^s$ remain finite as $\varepsilon \to 0$ and

$$\Lambda_{ij}{}^s (k, \varepsilon) = \Lambda_{ij}{}^s (k, 0) + O\,(|\ln \varepsilon|^{-1}),\ \varepsilon \to 0 \tag{3.2}$$

It follows from (3.1) that in the case of fluid spheres coefficients Λ_{ij} have no singularities as $\varepsilon \to 0$ (unlike in the axisymmetric case, see /14/). Moreover, numerical computations show that for fixed k, λ_1, and λ_2

$$\Lambda_{ij}\,(k, \varepsilon, \lambda_1, \lambda_2) = \Lambda_{ij}\,(k, 0, \lambda_1, \lambda_2) + O\,(\varepsilon) \tag{3.3}$$

as $\varepsilon \to 0$. By virtue of (3.2) relation (3.3) is inhomogeneous with respect to λ_1 and λ_2 when $\lambda_1, \lambda_2 \to \infty$.

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